Contest # 5

Problem 5-1

The solutions are -1, -2, and -3. The sum of their reciprocals is $-1 - \frac{1}{2} - \frac{1}{3} = \frac{-11}{6}$.

Problem 5-2

In any triangle, the length of the longest side must be less than the sum of the lengths of the other two sides, so 2020 + n < 2020 - n + 2020, and n < 1010. The largest possible integer value of n is 1009 (and such a triangle does exist).

Problem 5-3

This is basically a pigeon hole question. If there are 100 students and the only mode is 0, then we must make sure that no score occurs more often than a score of 0. Divide 100 by 16 (the number of possible scores) to get 6 (remainder 4). Thus, if each score is earned at most 7 times, we'd have 4 scores of 7 and 12 scores of 6. This doesn't allow for only one mode. If one score occurs 8 times, 2 scores occur 7 times, and the remaining 13 scores occur 6 times, we'd have accounted for all 100 students, and the only mode would be the score that occurred 8 times.

Problem 5-4

The vertex angles of the two isosceles triangles are supplementary, so the 4 base angles of the isosceles triangles have a sum of 180°. Reflecting the smaller isosceles triangle across the common base creates two congruent right triangles with legs of length 5 and 12. The two altitudes together make up the common hypotenuse of the two right triangles. Its length is 13.

Problem 5-5

Clearly, the numbers in question must have 3 digits each. Call one of them ABC. Call the other CBA. Since A × C ends in a 5, one of the numbers A or C—let’s say its A—must be 5. Since 92,565 ÷ 500 < 200, C = 1. To determine the value of B we notice that the 6 in the product is the last digit of 5B + B = 6B, so B = 1 or B = 6. All that remains is to test the 2 possibilities, 165 & 561 (which work).

Problem 5-6

In a geometric sequence with common ratio r and first term a, the first 3 terms are a, ar, and ar². We are told that a + 4, ar, and ar² form an arithmetic sequence, so r ≠ 1 and ar² - ar = ar - (a + 4) = ar - a - 4. This leads to ar² - 2ar + 1 = -4, so ar = -4 / (r - 1)². Since a and r are integers, (r - 1)² is a positive divisor of 4. The equation (r - 1)² = 2 has no rational solution, so the non-zero solutions of (r - 1)² = 1 or 4 are the only possibilities. From these two equations, we find that r = 2, 3, or -1. The three corresponding (a,b,c) triples are: (-4,-8,-16), (-1,-3,-9), (-1,1,-1).