Radical Expressions

A radical expression is one which contains a root (square root, cube root, etc.)

- **index (root)**: \( \sqrt{a} R \)radicand
- \( a \) is a positive integer > 1
- \( R \) is a real number

Rewriting Radicals

Radical can be written many ways. They can be written with a fractional exponent or they can be with radical signs.

For example: \( \sqrt{4^2} \) can be written as \( (\sqrt{4})^2 \) or as \( \frac{2}{3} \). When written as a fraction, the numerator represent the power and the denominator represents the root, so \( \frac{3}{5} \) can be written as \( \sqrt{5^3} \) or \( (\sqrt{5})^3 \).

Rewriting radicals can also help us simplify them. For example \( \frac{3}{5} = \sqrt{5^3} = \sqrt{5 \times 5 \times 5} = 5\sqrt{5} \). (Remember that the root tell you the number of constants and variables that you must have in order to remove 1. Since there were three 5’s under the radical, and we’re looking for the square root, we can take one out for 2 of the 5, leaving 1 still under the radical.)

If there are variables inside the radical we can still rewrite using exponential notation or as a radical expression. For example:

\[
\sqrt[3]{2^2} = 2^{\frac{2}{3}} \]

If it’s written in exponential notation, then the fractions must all have a common denominator, to be written as a radical expression. For example, in order to write \( 5^2 x^3 y^5 \) as a radical expression, we must rewrite each of the terms so that the exponent has a common denominator, which in this case would be 4, so \( 5^2 x^4 y = 5^\frac{2}{4} x^\frac{4}{4} y^\frac{5}{4} \). Now that they have the same denominator, we can write it as \( \sqrt[4]{5^2 x^3 y^4} \) or \( y \left( \sqrt[4]{5^2 x^3} \right) \).

Your Turn: Rewrite each of the following. The first one is done for you.

<table>
<thead>
<tr>
<th>Example</th>
<th>((3ab)^\frac{4}{5})</th>
<th>(\sqrt[5]{(3ab)^4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>(\sqrt[5]{10})</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td></td>
<td>(\sqrt[5]{5})</td>
</tr>
<tr>
<td>3.</td>
<td>(\frac{4}{x^3})</td>
<td>(\sqrt[5]{4x^3})</td>
</tr>
<tr>
<td>4.</td>
<td>(\sqrt[5]{4ab^5})</td>
<td></td>
</tr>
</tbody>
</table>
Simplifying Radical Expressions

There are several ways that you can simplify a rational expression. You can find the largest perfect index factor of the Radicand, for example: If we want to simplify $\sqrt{32}$, we can write 32 as $16 \times 2$, so $\sqrt{32} = \sqrt{16} \times \sqrt{2}$, the square root of 16 is 2, so $\sqrt{32} = \sqrt{16} \times 2 = 4\sqrt{2}$. Now let’s look at another example: $\sqrt[4]{32a^5b^3}$. $2^4 = 32$, so we can write the problem as $\sqrt[4]{2^4a^5b^3}$. Now let’s rewrite this one using fractional exponents. $\sqrt[4]{2^4a^5b^3} = \frac{4}{4}a^\frac{5}{4}b^\frac{3}{4}$. $\frac{4}{4} = 1$ and $\frac{8}{4} = 2$ and $\frac{5}{4} = \frac{1}{4}$, so we can rewrite $2^\frac{4}{4}a^\frac{5}{4}b^\frac{3}{4}$ as $2ab^\frac{3}{4}$. So $\sqrt[4]{32a^5b^3}$ can be simplified to $2ab\sqrt[4]{b}$.

Suppose that you don’t know the perfect index factor, for example 96. You can find all the prime factors of the number by either creating a factor tree, or by dividing.

Factor Tree

```
  96
 /   \
 2   48
  \   \n   4  12
   \  \n    2  6
     \  \n      2  3
```

So the factors of 96 are $2 \times 2 \times 2 \times 2 \times 2 \times 3$

Dividing – start with the first prime, and continue dividing by primes.

```
  96
/   \
2\ | 96
 \ | 48
  | 24
   | 212
    | 16
     | 313
      | 1
```

Again, the factors of 96 are $2 \times 2 \times 2 \times 2 \times 2 \times 3$

So if we want to find $\sqrt[4]{96a^5b^3}$, we can write the expanded form of what under the radical.

$\sqrt[4]{96a^5b^3} = \sqrt[4]{2^2 \times 2 \times 2 \times 2 \times 3 \times a \times a \times a \times b \times b \times b}$

Remembering back to Algebra I, if we have the number of constants or variables as the index/root, we can take one out. Since our root is 4, we have 4 2’s and 4 a’s and 4 b’s so we can take one of each out, so $\sqrt[4]{96a^5b^3} = 2ab\sqrt[4]{3 \times 2 \times 3 \times a \times a \times b \times b \times b}$.

Your Turn: Simplify each of the following:

1. $\sqrt[4]{144a^6b^{20}}$
2. $\sqrt[3]{-125x^{12}y^6}$
3. $\sqrt[4]{64x^{18}y^{12}}$

Rationalizing Denominators

In order to rationalize the denominator, we need to get rid of all radicals that are in the denominator.

Step 1: Multiply the numerator and denominator by a radical that will get rid of the radical in the denominator.

Step 2: Make sure all radicals are simplified.

Step 3: Simplify the fraction if needed.
Example 1: Rationalize the denominator \( \frac{3}{\sqrt{6}} \)

**Step 1:** Multiply numerator and denominator by a radical that will get rid of the radical in the denominator. So in this case we will multiply both the numerator and denominator by \( \sqrt{6} \).

\[
\frac{3 \sqrt{6}}{\sqrt{6} \sqrt{6}} = \frac{3 \sqrt{6}}{6}
\]

**Step 2:** Make sure all radicals are simplified. \( \sqrt{6} \) is simplified.

**Step 3:** Simplify the fraction if needed. (Both the numerator and denominator can be divided by 3.)

\[
\frac{\sqrt{6}}{2}
\]

**Your Turn:** Simplify each of the following:

1. \( \frac{35}{3\sqrt{15}} \)
2. \( \frac{7}{\sqrt{3}a} \)
3. \( \frac{8}{3\sqrt{2}} \)

**Rationalizing the Denominator**

(with two terms)

**Step 1:** Find the conjugate of the denominator. You find the conjugate of a binomial by changing the sign between the two terms, but keep order of the terms. \( a + b \) and \( a - b \) are conjugates of each other.

**Step 2:** Multiply the numerator and the denominator of the fraction of the conjugate found in Step 1.

**Step 3:** Make sure all radicals are simplified.

**Step 4:** Simplify the fraction if needed.

Example 1: Rationalize the denominator \( \frac{4}{x + \sqrt{6}} \)

**Step 1:** Find the conjugate of the denominator.

**Step 2:** Multiply numerator and denominator of the fraction of the conjugate found in Step 1. (REMINDER: When multiplying binomials that are conjugates of each other, you just need to multiply the first terms \( x(x) = x^2 \) and multiply the last terms, \( \sqrt{6}(\sqrt{6}) = 6 \).

\[
\frac{4 \left( x - \sqrt{6} \right)}{x + \sqrt{6} \left( x - \sqrt{6} \right)} \]

\[
\frac{4 \left( x - \sqrt{6} \right)}{x^2 - 6}
\]

Let’s try another example:

Example 1: Rationalize the denominator \( \sqrt{2 + \sqrt{3}} \)

**Step 1:** Find the conjugate of the denominator.

\[
\sqrt{5 + \sqrt{6}}
\]
Step 2: Multiply numerator and denominator of the fraction of the conjugate found in Step 1. (REMINDER: When multiplying binomials that are conjugates of each other, you just need to multiply the first terms $x(x) = x^2$ and multiply the last terms, $\sqrt{6}(\sqrt{6}) = 6$.

\[
\frac{\sqrt{2} + \sqrt{3} \left( \frac{\sqrt{5} + \sqrt{6}}{\sqrt{5} - \sqrt{6}} \right)}{\frac{\sqrt{10} + \sqrt{12} + \sqrt{15} + \sqrt{18}}{5 - 6}} \quad \frac{\sqrt{10} + 2\sqrt{3} + \sqrt{15} + 3\sqrt{2}}{-1} - \left(\sqrt{10} - 2\sqrt{3} + \sqrt{15} + 3\sqrt{2}\right)
\]

Step 3: Make sure all radicals are simplified. $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$ and $\sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$

Step 4: Simplify the fraction if needed.

YOUR TURN: Rationalize the denominator.

1. $-\frac{1}{2 + \sqrt{3}}$
2. $-\frac{2}{\sqrt{2} - \sqrt{5}}$
3. $\frac{4}{4 + \sqrt{2}}$
4. $\frac{\sqrt{5} + \sqrt{3}}{\sqrt{3} - 4}$
5. $\frac{5}{7 - 4\sqrt{3}}$
6. $\frac{\sqrt{6}}{\sqrt{11} - 2\sqrt{3}}$
Rational Equations

A rational equation is one that contains fractions with xs in the numerator, denominator or both. A rational equation looks like this: \( \frac{8}{x-2} - \frac{4}{x+2} = \frac{3}{x^3-1} \), so yes, in order to solve this equation you will need to find a common denominator.

The steps to solve a rational equation are:

- Find the common denominator
- Multiply everything by the common denominator
- Simplify
- Check your answer(s) to make sure there isn’t an extraneous solution.

Example 1:

To solve this problem, we need to find a common denominator. **A common denominator can always be found by multiplying all of the denominators together.**

Because the terms on the left hand side of the fraction are the same our denominator will be \((x - 2)(x + 3)\).

The first term on the left hand side of the equation already has \((x + 3)\), we need to multiply the first term by \((\frac{x - 2}{x + 3})\).

The second term on the left hand side already has the \((x - 2)\) in the denominator so we need to multiply that term by \((\frac{x + 3}{x - 2})\). On the right hand side of the equation the denominator already has \(x + 3\), so we need to multiply this term by \((\frac{x - 2}{x + 3})\).

Now we will simplify each term. **DO NOT MULTIPLY THE DENOMINATORS** and since on the left hand side the denominators are the same, we can put the numerators over one denominator.

Now, solve the denominators are the same, we can essentially drop them. So you will just need to take the numerators and set them equal to each other and solve.

Now we need to check our answer. There are two reasons to check our answer, one, we want to make sure that we didn’t make a mistake, and secondly, because sometimes we get an extraneous solution. To check, put your value(s) in for all x’s and solve. Since we have \(2 = 2\), and we didn’t divide by zero anywhere, our answer \(x - 2\) is correct.
Now let’s look at an example where there is an extraneous solution. (Note the problem will still be done the exact way. It will be the check, where we determine that there is an extraneous solution.

Example 2:

\[
\frac{1}{a+2} + \frac{1}{a-2} = \frac{4}{a^2 - 4}
\]

This problem is just a little bit different from the problem above. In this problem the first thing that you should notice is that there is a squared term in the denominator. When there is a squared term in the denominator, we should try to factor if possible. It usually makes the problem a little easier. I also know that \(a^2 - 4\) is the difference of two squares so the factors are \((a - 2)(a + 2)\).

Now, we will rewrite the problem.

We can now see that the common denominator will be \((a - 2)(a + 2)\) So we will need to multiply the first term in the expression on the left hand side by \(\frac{a-2}{a-2}\) and the second term in the expression on the left hand side by \(\frac{a+2}{a+2}\). Notice that because the right hand side of the equation already had both values, we do not need to multiply it by anything. Now, just as we did above, we will simplify the expression on the left hand side and put both of the terms over the same denominator. Again, since the denominators are now the same, we just need to work with the numerators. So writing the numerators, simplifying and solving, as we did above we find that \(a = 2\).

Now we need to check our work. So we will substitute 2 back in the original equation everywhere there is an \(a\).

We can see that when we substitute 2 in for \(a\), the second term on the left hand side becomes 1 divided by 0 and the term on the left hand side become 4 divided by 0. We cannot divide by zero so this equation does not have a solution. The answer \(a = 2\) is considered an extraneous root, because it is an invalid solution that occurred when the rational equation was manipulated.
Your Turn

Solve each of the following. Make sure you check your work and identify any extraneous solution that may occur.

<table>
<thead>
<tr>
<th>Work</th>
<th>Check</th>
</tr>
</thead>
</table>
| 1. \[
\frac{x+3}{x-4} - \frac{x-5}{x+4} = x
\] |       |
| 2. \[
\frac{2}{x^2-1} - \frac{1}{x^2+x} = \frac{3}{x}
\] |       |
| 3. \[
\frac{2}{x^2-1} - \frac{1}{x-1} = \frac{1}{2}
\] |       |
<table>
<thead>
<tr>
<th>Work</th>
<th>Check</th>
</tr>
</thead>
<tbody>
<tr>
<td>4. ( \frac{x}{x^2 - 1} - \frac{1}{x - 1} = \frac{1}{2} )</td>
<td></td>
</tr>
<tr>
<td>5. ( \frac{3}{x - 5} \cdot \frac{20}{x^2 - 25} = \frac{2}{x + 5} )</td>
<td></td>
</tr>
<tr>
<td>6. ( \frac{5}{x - 2} = 7 - \frac{10}{x + 2} )</td>
<td></td>
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</tbody>
</table>