Multiplying and Dividing Rational Expressions

Vocabulary:

**Binomial** - An algebraic expression of two terms (ex: \((x - 2)\))

**Trinomial** - An algebraic expression of three terms. (ex: \(5x^2 - 3x + 6\))

**Coefficient** - The number in front of the variable. (For ex. In \(7x^2, 7\) is the coefficient)

**Reciprocal** - or multiplicative inverse, is simply one of a pair of numbers that, when multiplied together, equal 1. If you can reduce the number to a fraction, finding the reciprocal is simply a matter of transposing the numerator and the denominator. ... The reciprocal of 7 is \(\frac{1}{7}\) because \(7 \times \frac{1}{7} = 7\).

In order to begin these lesson let’s establish our prior knowledge with 4 basic examples of factoring. (Please use notes from packet 2 for reference)

**GOLDEN RULE: ALWAYS CHECK FOR GCF FACTORING FIRST!**

**Trinomial with leading coefficient of 1.** (Remember always 3 terms here)

\[x^2 - 6x - 16\]

No GCF here! We need two numbers that add to -6 and multiply to -16. One integer will be positive and the other will be negative (to multiply to a negative). Since they add to a negative the larger (absolute) value must be the negative. \(-8 + 2 = -6\) Check \(-8 \cdot 2 = -16\) Check

Solution=\((x - 8)(x + 2)\)

**Difference of Perfect squares.** (Remember 2 two terms here that must be perfect squares)

\[25n^2 - 1\]

No GCF here! We take the \(\sqrt{\text{sqrt}()}\) of each term and split the same square into conjugate binomials. Remember in order to get that -1 back our binomial will have a positive and negative (hence conjugate).

Solution=\((5n + 1)(5n - 1)\)

**Trinomial with leading coefficient of greater than 1.** (Remember always 3 terms here & leading coefficient >1)

\[4j^2 + 12j + 9\]

No GCF here! Only thing different here is we need to start by multiplying the 4 and the 9 we get 36. Now what two integers add to 12 but multiply to 36? Positive 6 and positive 6 of course!!

\(4(j^2 + 3j) + (6j + 9)\)

I prefer to put the 8 first can be done with the 4 first!
In order to finish this factoring we must pull the GCF from the first binomial. The GCF is 2j! The 2\(^{nd}\) binomial must match the first in order for this to be done correctly! Since the 2\(^{nd}\) binomial is \((2j + 3)\) a 3 was factored out represented here below.

\[2j(2j + 3) + 3(2j + 3)\]

Since we got the binomial to repeat that comes down as a factor and for the other factor we must bring down the remaining terms. Pictured here below.
This does not happen every time!! It just so happens on this example that our repeating binomial and the remaining terms matched! Since we have the same binomial multiplied by the same exact binomial this can be represented with a square picture here.

Solution = \((2j + 3)^2\)

Special Case Factoring: GCF first and trinomial 2\(^{nd}\).

Check for a GCF first. We have one!! The GCF is 4. Factor it out!

\[4d^2 + 12d - 16\]

Now look at the trinomial inside the parenthesis. That can be factored!!! What adds to 3 and multiplies to -4? Positive 4 and negative 1!! Factor it!!!

Solution = \(4(d + 4)(d - 1)\)

These are some examples that will encompass the skill and processes necessary for today’s lesson.

A. Simplify  
\[\frac{x^2 - 2x - 24}{2x^3 + 6x^2 - 8x}\]

B. State when original expression is undefined.

In order to complete part A we will need to factor and simplify. For part b it’s imperative that we look at the full factored expression before fully simplifying it. Looking at the fully factored denominator we set all factors in the denominator equal to zero in order to find when the expression in undefined. Remember an expression is undefined when the denominator is equal to zero.

Let us focus on the numerator first it has no GCF, what adds to -2 and multiplies to -24? Negative 6 and positive 4!! Factor it!!

Now the denominator, it has a GCF 2x!!! Factor it out!!

\[\frac{(x - 6)(x + 4)}{2x(x + 4)(x - 4)}\]

The denominator has a trinomial (in the parenthesis) that can be factored further! What adds to 3 and multiplies to -4? Positive 4 and negative 1, factor it out!!

\[\frac{(x - 6)(x + 4)}{2x(x + 4)(x - 1)}\]

Before we simplify set all factors in the denominator equal to zero to complete part B.

\[\begin{align*}
x - 6 &= 0 & \Rightarrow & & x &= 6 \\
x + 4 &= 0 & \Rightarrow & & x &= -4 \\
2x &= 0 & \Rightarrow & & x &= 0
\end{align*}\]

Note- If it’s just a constant in front of the factors it does not count as a factor because it does not contain a variable. 2x is a factor because of the variable.

Part B solution = values of \(x\) that make this expression undefined are \(x = \{-4, 0, 1\}\).

Now back to part Part A the factor of \((x + 4)\) is in both the numerator and the denominator meaning they cancel through division. Leaving our fully simplified version of this expression as

Solution = \(\frac{x - 6}{2x(x - 1)}\)

A. Simplify  
\[\frac{x^2 + 2x + 1}{4x^2 + 3x - 1}\]

B. State when the original expression is undefined.

Factoring the numerator we need two integers that multiply to 1 and add to 2. Positive 1 and positive 1!!

\[\frac{(x + 1)(x + 1)}{4x^2 + 3x - 1}\]
Note we could have represented this numerator with a square instead but left it this way to more easily see the simplification.

Factoring the denominator we need to multiply the 4 and the negative 1 first giving us \(-4\). We need two integers that multiply to \(-4\) and add to 3. 4 and -1!! Factor!!

Now just focusing on the denominator we have

\[(4x^2 + 4x) + (-x - 1)\]

Take the GCF in the 1st binomial and factor it out! GCF \(= 4x\). The 1st resulting binomial will match the 2nd binomial in order to factor this!! We have to factor out a negative 1 from the 2nd binomial in order to get it to match.

\[4x(x + 1) - 1(x+1)\]

Now the repeating binomial is a factor and the two remaining terms for the other factor leaving us with this for our denominator.

\[(4x - 1)(x + 1)\]

Before placing back in our denominator and simplifying let us complete part b by setting these two factor equal to zero and determining when this expression is undefined.

\[
\begin{align*}
4x - 1 &= 0 \\
x &= \frac{1}{4}
\end{align*}
\]

\[
\begin{align*}
x + 1 &= 0 \\
x &= -1
\end{align*}
\]

**Part B Solution:** This expression is undefined when \(x = \{\frac{1}{4}, -1\}\)

Now for Part A let’s place our factored denominator in and start simplifying!

\[
\frac{(x + 1)(x + 1)}{(4x - 1)(x + 1)}
\]

Factors of \((x + 1)\) are both in the numerator and denominator, we can cancel Them!!! Giving us our fully simplified answer below.

\[
\text{Solution for Part A} = \frac{(x+1)}{(4x-1)}
\]

**Special Case:** Factoring out a negative one to fully simplify:

\[
\begin{align*}
\frac{x^2 - 5x + 6}{-x + 2} &= \frac{(x-2)(x-3)}{-(x-2)} \\
&= \frac{(x-2)(x-3)}{-x+2} \\
&= -x + 3
\end{align*}
\]

**Solution**

\[\text{Hence} \quad \frac{x-a}{(a-x)} = -1, \text{ always}!!!\]

Now let’s look at this example embedded in a larger more test type question.

Simplify: \(\frac{(6x^2 - 5xy)(x + 2y)}{(x+y)(5y - 6x)}\)
Factor out the GCF from the left binomial in the numerator.

\[
\frac{x(6x - 5y)(x + 2y)}{(x + y)(5y - 6x)}
\]

Some student may see the negative one rule here and just cancel leaving a negative 1 behind. Here I will show the factoring out of the negative one first.

\[
\frac{x(-1)(5y - 6x)(x + 2y)}{(x + y)(5y - 6x)}
\]

Now we can clearly see this cancellation, cancel it and this will leave us with our full simplified form.

Solution = \(-\frac{x(x + 2y)}{(x + y)}\)

Simplify this expression: \(\frac{3x}{8y} \cdot \frac{12x^2y}{9xy^3}\)

Multiply straight across (I prefer this way, you could break it down into prime factors but I find that process to be redundant). Numerators to numerators and denominators to denominators.

\[
\frac{36x^3y}{72xy^4}
\]

Reduce your fraction and simplify your exponents!

Solution = \(\frac{x^2}{2y^3}\)

Simplify this expression: \(\frac{10d^5}{6cd} \div \frac{30c^3d^2}{4c}\)

Same type of problem just different operation. The way we can make it exactly the same is by flipping our 2nd fraction. When we do the reciprocal we can now make the operation multiplication.

\[
\frac{10d^5}{6cd} \cdot \frac{4c}{30c^3d^2}
\]

Multiply straight across, Numerators to numerators and denominators to denominators!

Note - Also you can simplify coefficients before multiplying when they start to get larger I actually prefer to do it that way!!

\[
\frac{40cd^5}{180c^4d^3}
\]

Reduce your fraction and simplify your exponents! (Hint : The GCF for the whole numbers is 20)

Solution = \(\frac{2d^2}{9c^3}\)

Simplify: \(\frac{3x}{x - y} \div \frac{6xy}{4x^2 - 4y^2}\) Instead of dividing, multiply by the reciprocal!

\[
\frac{3x}{x - y} \cdot \frac{4x^2 - 4y^2}{6xy}
\]

The left fraction is simplest form but the right numerator has a GCF!

\[
\frac{3x}{x - y} \cdot \frac{4(x^2 - y^2)}{6xy}
\]

The right numerator can be factored further by using difference of perfect squares!

\[
\frac{3x}{(x - y)} \cdot \frac{4(x + y)(x - y)}{6xy}
\]

There is an identical factor in the numerator and denominator, cancel it!
\[ \frac{12x(x + y)}{6xy} \]

Reduce fraction to a whole number (divides evenly) and cancel your x!

**Solution:** \[ \frac{2(x+y)}{y} \]

**Real World Application of these types of problems:**

Aidrian is building a rectangular deck. If the length is \( \frac{y^2+8y+1}{y-6} \) and the width is \( \frac{y^2-9y+18}{y^2-9} \). Write and simplify an expression that represents the area of the deck.

Area of a rectangle = Length Times Width or \( A = L \cdot W \). Apply this formula.

\[
\text{Area} = \frac{y^2+8y+15}{y-6} \cdot \frac{y^2-9y+1}{y^2-9}
\]

Factor and Simplify!! We will use trinomial factoring and difference of perfect squares. This is equal to the area.

\[
\frac{(y + 5)(y + 3)}{(y - 6)} \cdot \frac{(y - 6)(y - 3)}{(y + 3)(y - 3)}
\]

**Solution:** \( A = (y + 5) \)

Note all 3 binomials from the denominator cancelled leaving behind one lone binomial in the numerator!

**Reminder of negative exponents:** \( (\frac{x}{y})^{-1} = \frac{y}{x} \)

Another example: \( (\frac{20xy^3}{9x^4y^3})^{-1} = \frac{9x^4y^3}{20xy^3} \)

**Reminder of complex fractions.** Don’t divide multiply by the reciprocal!

\[
\frac{30xy}{8ab} \cdot \frac{2xy}{15ab}
\]

Can be represented like \( \frac{30xy}{8ab} \cdot \frac{2xy}{15ab} \) and then you can simplify! **Solution:** \( \frac{x^2y^2}{2a^2b^2} \)

**Reminder of exponential rules here!**

### Exponent Properties Review

- **Product Rule:** \( x^3 \cdot x^6 = x^9 \)
- **Quotient Rule:** \( \frac{x^6}{x^3} = x^4 \)
- **Power Rule:** \( (x^2)^3 = x^6 \)
- **Negative Exponent Rule:** \( x^{-3} = \frac{1}{x^3} \)
YOUR TURN: Simplify this expression: Also state when the original expression is undefined (For #'s 1-2) All work needs to be done on your paper. Keep your work with your packet.

1. \( \frac{x^2(x + 2)(x - 4)}{6x(x^2 + x - 20)} \)
2. \( \frac{3y(y - 8)(y^2 + 3y + 2)}{2y(y - 4)(y + 2)} \)

Simplify Each Expression: (for #'s 3-12)

3. \( \frac{(x - 4)(x^2 + 2x - 48)}{(36 - x^2)(x^2 + 4x - 32)} \)
4. \( \frac{x^2 - 9y^2}{2x + 6y} \)
5. \( \frac{3ac^3f^3}{8a^2bcf^4} \cdot \frac{12ab^2c}{18ab^3c^2f} \)
6. \( \frac{14x^2y^2z^3}{21w^4x^2z} \cdot \frac{7wxyz}{12w^2y^3z} \)
7. \( \frac{x - y}{x^2 - 6y} \)
8. \( \frac{2x^2 + 7x + 3}{(2x^2 - 15x + 7)^{-1}} \cdot \frac{9 - x^2}{x^2 - 4x - 21} \)

9. The producer of hair care products wants to compare the areas of two different containers. For some measure x, the area of the first can be represented by \( \frac{16b^2 + 40b + 25}{3b^2 - 10b - b^5} \). The area of the 2nd container is represented, \( \frac{4b + 5}{b^5} \). Write and simplify and expression that represents the ratio of the 1st container to the area of the 2nd container.

10. Matt is building a rectangular patio. If \( \frac{y^2 + 19y + 84}{4y - 4} \) represents the length of the patio and \( \frac{2y - 2}{y^2 + 9y + 14} \) represents the width, write and simplify an expression that represents the area of the patio.

*11. Rodney graphs the function \( y = \frac{3(x - 1)(x + 2)(x - 4)}{(x + 2)(x - 4)(x - 1)} \) on a graphing calculator and claims the result is just a horizontal line. Is he correct? Justify your explanation.

Adding and Subtracting Rational Expressions Day 2

In mathematics, the lowest common denominator or least common denominator (abbreviated LCD) is the lowest common multiple of the denominators of a set of fractions. It simplifies adding, subtracting, and comparing fractions.

In order to add and subtract rational expressions we must be able to find the LCD.
An example of finding the LCD is pictured here below, note we needed to factor this one 1st in order to see it.

Let's look at a few examples!!

Simplify this expression: \( \frac{7a}{4b} + \frac{4c^2}{10} \)

The LCD here would be 20b. This means we have to multiply the left fraction by 5 in order to get our LCD (remember we're just changing the representation not the value, essentially when we multiply by \( \frac{5}{5} \) it's really just multiplying by one! We will need to multiply the right fraction by \( \frac{2b}{2b} \).

\[
\begin{align*}
\frac{7a}{4b} \cdot \frac{5}{5} + \frac{4c^2}{10} \cdot \frac{2b}{2b} \\
\frac{35a}{20b} + \frac{8c^2b}{20b} \\
\text{Solution} = \frac{35a + 8c^2b}{20b}
\end{align*}
\]

Simplify this expression: \( \frac{2x}{9y} - \frac{7y}{6z} \)

The LCD is 18yz. We need to multiply the left fraction by \( \frac{2x}{2x} \) and we need to multiply the right fraction by \( \frac{3y}{3y} \).

\[
\begin{align*}
\frac{2x}{9y} \cdot \frac{2z}{2z} - \frac{7y}{6z} \cdot \frac{3y}{3y} \\
\frac{4xz}{18yz} - \frac{21y^2}{18yz} \\
\text{Solution} = \frac{4xz - 21y^2}{18yz}
\end{align*}
\]

Now let's look at an example where factoring is required first!

Simplify this expression: \( \frac{2x+1}{x^2+2x-1} - \frac{7}{5x-15} \) Factor first!!

\[
\begin{align*}
\frac{2x+1}{(x-3)(x+5)} - \frac{7}{5(x-3)} \\
\frac{(2x+1)}{(x-3)(x+5)} \cdot \frac{5}{5} - \frac{7}{5(x-3)} \cdot \frac{(x+5)}{(x+5)} \\
\frac{10x+5 - (7x + 35)}{5(x-3)(x+5)} \\
\end{align*}
\]
Solution \(= \frac{3x - 30}{5(x - 3)(x + 5)}\)

Simplify this expression: \(\frac{3x}{4x^2 + 4} - \frac{2x^2}{x^4 - 1}\)

Factor!

\[\frac{3x}{4x^2 + 4} - \frac{2x^2}{x^4 - 1}\]

The LCD here is \((x^2 - 1)(x^2 + 1)\).

\[
\frac{3x}{4(x^2 + 1)} \cdot \frac{(x^2 - 1)}{(x^2 - 1)} - \frac{2x^2}{(x^2 - 1)(x^2 + 1)} \cdot \frac{4}{4}
\]

\[
= \frac{3x(x^2 - 1) - 8x^2}{4(x^2 + 1)(x^2 - 1)} - \frac{3x^3 - 8x^2 - 3x}{4(x^2 + 1)(x^2 - 1)}
\]

Solution \(= \frac{3x(x^2 - 1) - 8x^2}{4(x^2 + 1)(x^2 - 1)} - \frac{3x^3 - 8x^2 - 3x}{4(x^2 + 1)(x^2 - 1)}\)

Simplify complex fractions by using different LCD’s in the numerator and denominator.

Simplify this expression: \(\frac{x - 1}{x + y}\)

The LCD of the numerator is \(y\) and the LCD of the denominator is \(xy\).

\[
\frac{x}{y} + 1 \quad \frac{y}{y} - \frac{1}{y} \\
\frac{x + y}{y} \quad \frac{y - 1}{y} \\
\frac{y + x}{y} \quad \frac{y}{y} \\
\frac{y^2 - x}{xy} - \frac{x}{xy}
\]

Instead of dividing multiply by the reciprocal!!

\[
\frac{x + y}{y} \cdot \frac{xy}{y^2 - x}
\]

The \(y\) in the left denominator and in the right numerator cancel!!

\[
\frac{x(x + y)}{y^2 - x} = \frac{x^2 + xy}{y^2 - x} = \text{Solution}
\]

Directions: Simplify Each Expression for #’s (1-10) Do all work on your own paper.

1. \(\frac{3}{x} + \frac{5}{y}\)
2. \(\frac{3}{w - 3} - \frac{2}{w^2 - 9}\)
3. \(\frac{3t}{2 - x} + \frac{5}{x - 2}\)
4. \( \frac{n}{n - 3} + \frac{2n + 2}{n^2 - 2n - 3} \)

5. \( \frac{3}{y^2 + y - 12} - \frac{2}{y^2 + 6y + 8} \)

6. \( \frac{4}{x + 5} + \frac{9}{x - 6} \)

7. \( \frac{5}{x - 6} - \frac{8}{x + 5} \)

7. The electric potential between two electrons is given by the formula that has the form \( \frac{1}{r} + \frac{1}{1-r} \). Simplify this expression.

8. Timmy is building a rectangular fence around his yard. In terms of x and y, the width is \( \frac{x}{y-2} \) feet and the length is \( \frac{3+x}{y} \) feet.

Part A: Write an expression in simplest form that can represent the amount of fencing he needs? (in terms of x and y)

Part B: Are there any restrictions on the variables?