Name: ______________________________________ Date: ______________________

Calculus 3: Summer Assignment – PreCalculus Review

**Section 1:**

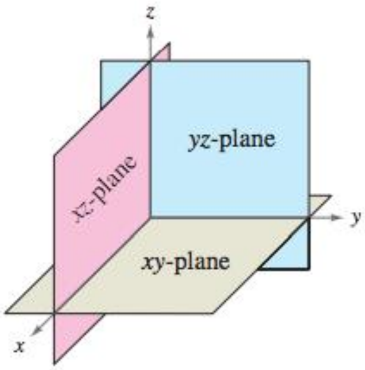



Figure 10.1

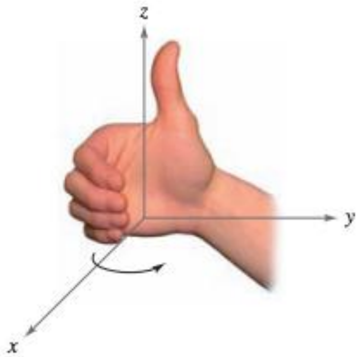



Figure 10.2

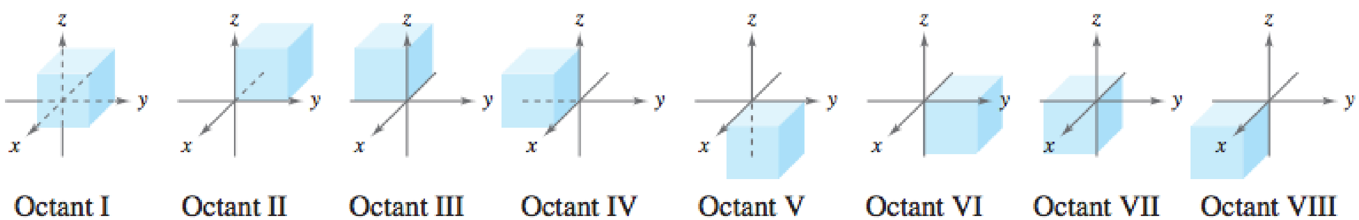
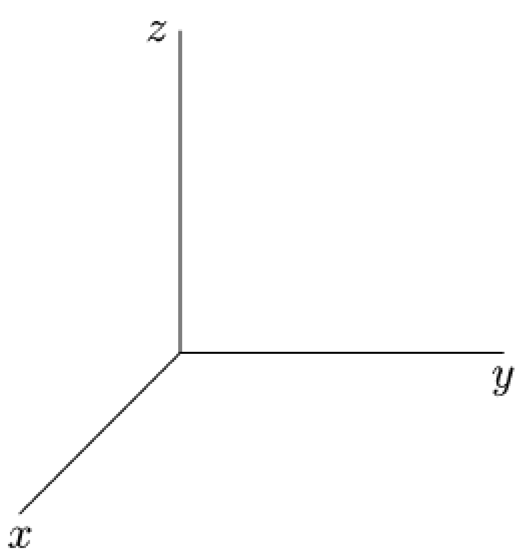



Figure 10.3

## Example 1 Plotting Points in Space

Plot each point in space.

**a.** $(2, -3, 3)$ **b.** $(-2, 6, 2)$ **c.** $(1, 4, 0)$ **d.** $(2, 2, -3)$



**Distance Formula in Space**

The distance between the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the **Distance Formula in Space** is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

## Example 2 Finding the Distance Between Two Points in Space

Find the distance between $(0, 1, 3)$ and $(1, 4, -2)$.

**Midpoint Formula in Space**

The midpoint of the line segment joining the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ given by the **Midpoint Formula in Space** is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

## Example 3 Using the Midpoint Formula in Space

Find the midpoint of the line segment joining $(5, -2, 3)$ and $(0, 4, 4)$.

**Standard Equation of a Sphere**

The **standard equation of a sphere** with center $(h, k, j)$ and radius $r$ is given by

$$(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2.$$

## Example 4 Finding the Equation of a Sphere

Find the standard equation of the sphere with center $(2, 4, 3)$ and radius 3. Does this sphere intersect the $xy$-plane?

## Example 5 Finding the Center and Radius of a Sphere

Find the center and radius of the sphere given by

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0.$$

## Example 6 Finding a Trace of a Surface

Sketch the $xy$-trace of the sphere given by $(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 5^2$.

**Exploration**

Find the equation of the sphere that has the points $(3, -2, 6)$ and $(-1, 4, 2)$ as endpoints of a diameter. Explain how this problem gives you a chance to use all three formulas discussed so far in this section: the Distance Formula in Space, the Midpoint Formula in Space, and the standard equation of a sphere.

**Section 2:**

### Component Form of a Vector

The component form of the vector with initial point $P = (p_1, p_2)$ and terminal point $Q = (q_1, q_2)$ is given by

$$\overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}.$$

The **magnitude** (or length) of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{v_1^2 + v_2^2}.$$

If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v}$ is the zero vector $\mathbf{0}$.

## Unit Vectors

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given nonzero vector $\mathbf{v}$. To do this, you can divide $\mathbf{v}$ by its length to obtain

$$\mathbf{u} = \text{unit vector} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}.$$

Unit vector in direction of $\mathbf{v}$

### Example 1 Finding the Component Form of a Vector

Find the component form and magnitude of the vector $\mathbf{v}$ having initial point $(3, 4, 2)$ and terminal point $(3, 6, 4)$. Then find a unit vector in the direction of $\mathbf{v}$.

### Example 2 Finding the Component Form of a Vector

Find the component form and magnitude of the vector $\mathbf{v}$ that has initial point $(4, -7)$ and terminal point $(-1, 5)$.

### Example 4 Finding a Unit Vector

Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5 \rangle$ and verify that the result has a magnitude of 1.

**Definition of Vector Addition and Scalar Multiplication**

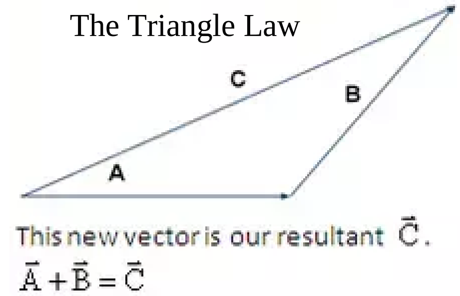
Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let $k$ be a scalar (a real number). Then the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle \qquad \text{Sum}$$

and the **scalar multiple** of $k$ times $\mathbf{u}$ is the vector

$$k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle. \qquad \text{Scalar multiple}$$

The Triangle Law

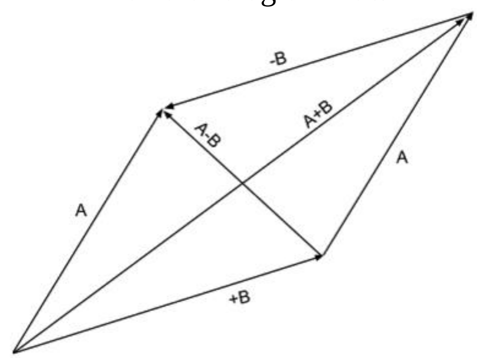



## Example 3 Vector Operations

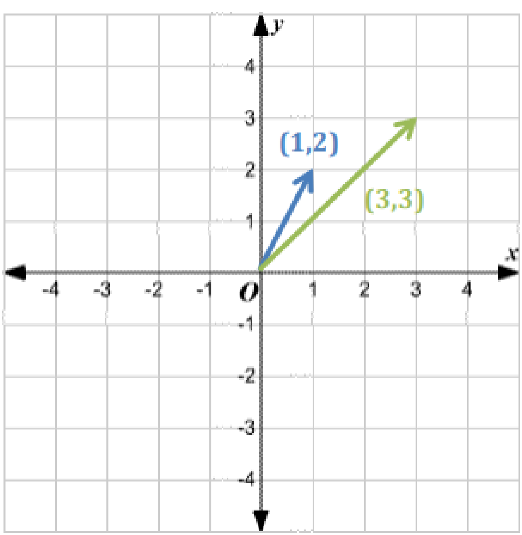
Let $\mathbf{v} = \langle -2, 5 \rangle$ and $\mathbf{w} = \langle 3, 4 \rangle$, and find each of the following vectors.

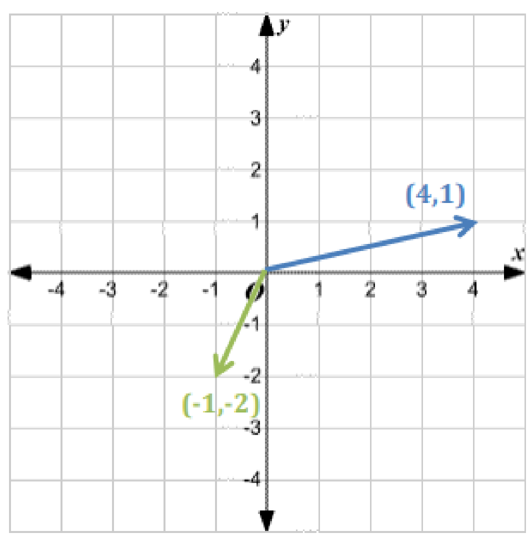
**a.** $2\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$ **d.** $2\mathbf{v} - 3\mathbf{w}$

The Parallelogram Law




Add the vectors. Sketch the resultant $\vec{R}$



## Linear Combination

$$\begin{aligned}\mathbf{v} &= \langle v_1, v_2\rangle \\ &= v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle \\ &= v_1\mathbf{i} + v_2\mathbf{j}\end{aligned}$$

The scalars $v_1$ and $v_2$ are called the **horizontal and vertical components of v,** respectively. The vector sum

$$v_1\mathbf{i} + v_2\mathbf{j}$$

is called a **linear combination** of the vectors **i** and **j**. Any vector in the plane can be written as a linear combination of the standard unit vectors **i** and **j**.

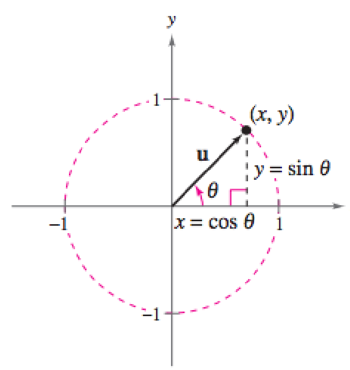
### Example 5 Writing a Linear Combination of Unit Vectors

Let **u** be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$. Write **u** as a linear combination of the standard unit vectors **i** and **j**.

## Direction Angles

If **u** is a *unit vector* such that $\theta$ is the angle (measured counterclockwise) from the positive $x$-axis to **u**, the terminal point of **u** lies on the unit circle and you have

$$\mathbf{u} = \langle x, y\rangle = \langle \cos\theta, \sin\theta\rangle = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$




### Example 8 Finding the Component Form of a Vector

Find the component form of the vector that represents the velocity of an airplane descending at a speed of 100 miles per hour at an angle of 30° below the horizontal, as shown in Figure 6.31.

**Section 3:**

**Definition of Dot Product**

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

## Example 1 Finding Dot Products

Find each dot product.

**a.** $\langle 4, 5 \rangle \cdot \langle 2, 3 \rangle$

**b.** $\langle 2, -1 \rangle \cdot \langle 1, 2 \rangle$

**c.** $\langle 0, 3 \rangle \cdot \langle 4, -2 \rangle$

## Example 2 Finding the Dot Product of Two Vectors

Find the dot product of $\langle 4, 0, 1 \rangle$ and $\langle -1, 3, 2 \rangle$.

## Example 3 Dot Product and Magnitude

The dot product of **u** with itself is 5. What is the magnitude of **u**?

**Angle Between Two Vectors** (See the proof on page 471.)

If $\theta$ is the angle between two nonzero vectors **u** and **v**, then

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}.$$

## Example 4 Finding the Angle Between Two Vectors

Find the angle between $\mathbf{u} = \langle 4, 3 \rangle$ and $\mathbf{v} = \langle 3, 5 \rangle$.

### Definition of Orthogonal Vectors

The vectors **u** and **v** are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

## Example 5 Determining Orthogonal Vectors

Are the vectors $\mathbf{u} = \langle 2, -3 \rangle$ and $\mathbf{v} = \langle 6, 4 \rangle$ orthogonal?

### Definition of Vector Components

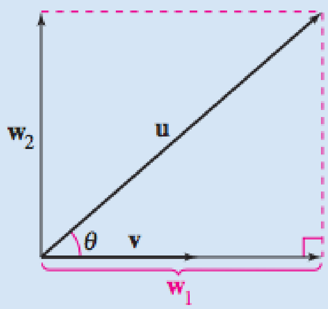
Let **u** and **v** be nonzero vectors such that

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

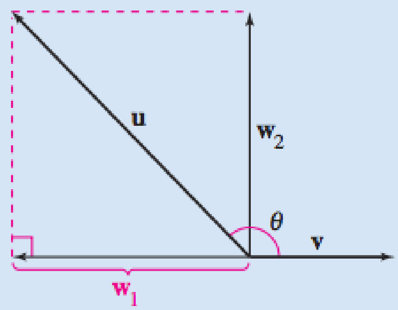
where $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal and $\mathbf{w}_1$ is parallel to (or a scalar multiple of) **v**, as shown in Figure 6.39. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are called **vector components** of **u**. The vector $\mathbf{w}_1$ is the **projection** of **u** onto **v** and is denoted by

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}.$$

The vector $\mathbf{w}_2$ is given by $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$.




*θ is acute.*




*θ is obtuse*

**Figure 6.39**

### Projection of u onto v

Let **u** and **v** be nonzero vectors. The projection of **u** onto **v** is given by

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

## Example 6 Decomposing a Vector into Components

Find the projection of $\mathbf{u} = \langle 3, -5 \rangle$ onto $\mathbf{v} = \langle 6, 2 \rangle$. Then write **u** as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}}\mathbf{u}$.

**Section 4:**

**Definition of Cross Product of Two Vectors in Space**

Let

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space. The **cross product** of **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

**Useful tool:**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Put **u** in Row 2.

Put **v** in Row 3.

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

## Example 1 Finding Cross Products

Given $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, find each cross product.

**a.** $\mathbf{u} \times \mathbf{v}$ **b.** $\mathbf{v} \times \mathbf{u}$ **c.** $\mathbf{v} \times \mathbf{v}$

## Exploration

Find each cross product. What can you conclude?

**a.** $\mathbf{i} \times \mathbf{j}$ **b.** $\mathbf{i} \times \mathbf{k}$ **c.** $\mathbf{j} \times \mathbf{k}$

## Example 2 Using the Cross Product

Find a unit vector that is orthogonal to both

$$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{v} = -3\mathbf{i} + 6\mathbf{j}.$$

**More Practice: find a unit vector orthogonal to u and v.**

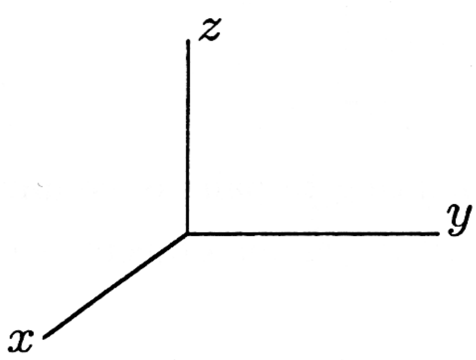
**27.** $\mathbf{u} = \langle 1, 2, 3 \rangle$
$\mathbf{v} = \langle 2, -3, 0 \rangle$

**29.** $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$
$\mathbf{v} = \mathbf{j} + \mathbf{k}$

## Example 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram. Then find the area of the parallelogram. Is the parallelogram a rectangle?

$A(5, 2, 0)$, $B(2, 6, 1)$, $C(2, 4, 7)$, $D(5, 0, 6)$



**Geometric Property of Triple Scalar Product**

The volume $V$ of a parallelepiped with vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

## Example 4 Volume by the Triple Scalar Product

Find the volume of the parallelepiped having

$$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}, \qquad \mathbf{v} = 2\mathbf{j} - 2\mathbf{k}, \qquad \text{and} \qquad \mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

as adjacent edges, as shown in Figure 10.25.

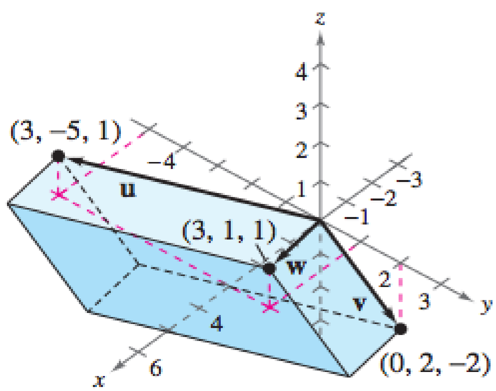



**Figure 10.25**