Contest # 2

Answers & Solutions

Problem 2-1
A square's sides have equal lengths, so \( x - 1 = 5 - x \), and \( x = 3 \). The square's area is \((3 - 1)^2 = 4\).

Problem 2-2
Since \(4x^2 + kxy + 4y^2 = (2x + 2y)^2 = 4x^2 + 8xy + 4y^2\), either \( k = 8 \) or \( k = -8 \). Their sum is \([0]\).

Problem 2-3
If the smaller triangle has side-lengths \( x \), \( x \), and \( y \), then the larger has side-lengths \( 2x \), \( 2x \), and \( y \). Since \( 2x + y = 18 \), while \( 4x + y = 28 \), it follows that \( x = 5 \) and the length of the base = \( y = 8\).

Problem 2-4
The least common multiple of \( n \) and \( 1000 = 2^3 \times 5^3 \) will be \( 2000 = 2^4 \times 5^3 \) if and only if \( n = 2^m \times 5^m \), where \( m \) is an integer, \( 0 \leq m \leq 3 \). The only possible values of \( n \) are \( 2^4 \times 5^0 \), \( 2^4 \times 5^1 \), \( 2^4 \times 5^2 \), and \( 2^4 \times 5^3 \). These four values of \( n \) are \([16, 80, 400, 2000]\).

Problem 2-5
Wherever a 20 appears in Jan’s five numbers, a 1 appears in Ann’s (and vice versa). The sum of two numbers in the same position for both Jan and Ann is 21, so the sum of all ten numbers (Jan’s first five plus Ann’s first five) is \( 5 \times 21 = 105 \). Since the sum of Jan’s five numbers is 66, the sum of the Ann’s five numbers is \( 105 - 66 = 39\).

Problem 2-6
Let’s try some. Which numbers can (and which can’t) be written as a sum of consecutive positive integers? 1; no; 2: no; 3 = 1 + 2; 4: no; 5 = 2 + 3; 6 = 1 + 2 + 3; 7 = 3 + 4; 8: no; 9 = 4 + 5; 10 = 1 + 2 + 3 + 4; 11 = 5 + 6; 12 = 3 + 4 + 5; 13 = 6 + 7; 14 = 2 + 3 + 4 + 5; 15 = 7 + 8; 16: no; 17 = 8 + 9; . . . . The positive integers that aren’t a sum of two or more consecutive positive integers are 1, 2, 4, 8, 16, . . . . These are all powers of 2, so the answer is probably the highest power of 2 less than 2018. That’s \( 2^{10} \) or \( 1024 \). Here’s a proof.

Theorem: A positive integer \( n \) can be written as a sum of two or more consecutive positive integers if and only if \( n \) has an odd divisor \( > 1 \).

Proof: For \( n > 0 \), if \( n = a + (a+1) + ... + (a+k) \) for some positive integers \( a \) and \( k \), the sum of the first and last terms is \( 2a+k \) and the number of terms is \( k+1 \), so \( n = (2a+k)(k+1)/2 \). If \( k \) is odd, then \( 2a+k \) is odd and \( k+1 \) is even; and if \( k \) is even, then \( k+1 \) is odd and \( 2a+k \) is even. Since both \( a \) and \( k \) are positive integers, both \( 2a+k \) and \( k+1 \) are greater than 1. Whichever of these is odd is an odd factor of \( n \) that is greater than 1. This proves that \( n \) is divisible by an odd number > 1. Let’s reverse direction. If \( n > 0 \) has an odd divisor > 1 then \( n = (2d+1)m \) for positive integers \( d \) and \( m \). Therefore, we can write \( n \) as the sum \((m-d) + (m-d+1) + ... + m + ... + (m+d-1) + (m+d) = (2d+1)m \). Unless \( m < d \), this is the sum of consecutive positive integers. If \( m < d \), cancel every negative term with its corresponding positive term, leaving a sum of consecutive terms which equals \((2d+1)m\). This indicated sum contains at least 2 positive integers, otherwise \( m-d = -(m+d-1) \), implying that \( m = 1/2 \). The only positive integers which cannot be expressed as a sum of two or more consecutive positive integers are those which have no odd divisor greater than 1. The only such positive integers are the powers of 2.