Before we begin with the Binomial Theorem, let’s look at Pascal’s Triangle. Pascal’s Triangle is a triangular array constructed by summing adjacent elements in the preceding rows. Pascal’s triangle contains the value of the binomial coefficient. If you can replicate Pascal’s triangle, you can raise a binomial to any power with very little work. Here is what the first six rows of Pascal’s Triangle looks like.

\[
\begin{array}{cccccc}
& & & 1 & & \\
& & 1 & & 1 & \\
& 1 & & 2 & & 1 \\
1 & & 3 & & 3 & & 1 \\
1 & & 4 & & 6 & & 4 & & 1 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\end{array}
\]

…

There is a pattern to the triangle, but it may be a little tricky to see. To construct Pascal’s triangle, we begin by placing a 1 at the top center. The next row down of the triangle is constructed by putting one, on the right and left hand side of the one on the top row. If you look the ones, you can see that they are on the outside of the triangle. Every row will start with a row and end with a row. If you look carefully, you can also see that the triangle is symmetric. If you draw a vertical line through the top one and folded it along the line, the right and left hand sides would line up.

\[
\begin{array}{cccccc}
& & & 1 & & \\
& & 1 & & 1 & \\
& 1 & & 2 & & 1 \\
1 & & 3 & & 3 & & 1 \\
1 & & 4 & & 6 & & 4 & & 1 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\end{array}
\]

So now the question becomes where to these numbers in the triangle come from?

If we look at the second and third rows, we can see that the second row has 2 ones, the third row looks like this: 1 2 1. The 2 is the sum of the two rows above. The numbers in the row below, is the sum of the numbers in the row above. So in the 4th row, the 4’s are coming from adding the 1 and the 3, and the 6 comes from adding the two 3’s. Knowing this you can expand this triangle, just by adding the numbers in the row above. Knowing this, find the numbers in the 6th and 7th row. (Note, when we talk about the 6th and 7th row, we’re talking about the first number to the right of the one.)

\[
\begin{array}{cccccc}
6^\text{th} \text{ Row} & & & & & \\
7^\text{th} \text{ Row} & & & & & \\
\end{array}
\]
Is this what you got?

<table>
<thead>
<tr>
<th>6th Row</th>
<th>7th Row</th>
</tr>
</thead>
<tbody>
<tr>
<td>1  6  15  20  15  6  1</td>
<td>1  7  21  35  35  21  7  1</td>
</tr>
</tbody>
</table>

There are some more patterns that exist in Pascal's triangle.

The first diagonal is, of course, just “1”s.
The next diagonal has the **Counting Number** (1, 2, 3, etc).
The third diagonal has the triangular numbers.

If you look at the sum of each row, they are the powers of 2.

Each line is also the powers (exponents) of 11.

\[
11^0 = 1 \\
11^1 = 11 \\
11^2 = 121 \\
11^3 = 1331
\]

As you can see the Pascal's triangle, can give us a lot of information. We are actually going to use Pascal's triangle, in looking at the Binomial Theorem. Before we start, we need to review a few more thing. You should remember that a **binomial** is a **polynomial with two term**. For example, \(2x + 7\) is a binomial. It has two terms, \(2x\) and 7.

You should also remember how to multiply a binomial.

\[
(x + y)^2 = (x + y)(x + y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2
\]

If you want to raise the binomial to the third power, \((x + y)^3\), you can expand the problem to be \((x + y)(x + y)(x + y)\), but you know that \((x + y)(x + y) = x^2 + 2xy + y^2\), so you need to use the distributive property to find the answer to \((x + y)(x^2 + 2xy + y^2)\).

**Simplify the following problems and circle your final answer. Use your answer from questions 1 to help you answer questions 2 and use your answer for question 2 to help you answer question 3.**

1. Expand: \((x + y)^3 = \)
2. Expand: \((x + y)^4 = \)

3. Expand: \((x + y)^5 = \)

4. Compare your answers to questions 1 – 3 with Pascal’s Triangle. Do you see any similarities or differences?

If we expand \((x + y)^5\), we will get \(x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\) and if we expand \((x + y)^6\), we will get \(x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6\). In order to expand these binomials, I did not do any multiplication. I went to Pascal’s Triangle and looked at the row where the second terms was a 6. The numbers in my expansion became the coefficients in my answer. Then I started with the first variable, \(x\), and raised it to the power that I wanted to expand my binomial to. From there the exponent on my first variable decreased as I moved from term to term. Beginning with the second term, I introduced my second variable, and it increased until I got to the final term a “1”, which is invisible and my second term had the power that I was raising the binomial to. Now you’re going to try it. Expand \((a + b)^7\), by filling in the blanks below.

\[(a + b)^7 = a^7 + 7a^6b + \_ a^5b^2 + \_ a^4b^3 + \_ a^3b^4 + \_ a^2b^5 + \_ ab^6 + b^7\]

(If you go to the bottom of page 6, you can check your answer.)

So now that you know that you can use Pascal’s Triangle, to expand a binomial raised to a power, what happens if you are subtracting the two terms instead of adding. Let’s look at \((x - y)^2 = (x - y)(x - y) = x^2 - 2xy + y^2\). So how can we use Pascal’s triangle? In order to use Pascal’s Triangle, you will need to rewrite the binomial. \((a - b)^4 = (a + (-b))^4\). Now we can use Pascal’s Triangle.

\[(a - b)^4 = (a + (-b))^4 = a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^4\]. Since we know that a negative raised to an even power will be positive and a negative raised to odd power will be negative, we can see that our final answer will be: \((a - b)^4 = a^4 - 4a^2b + 6a^3b^2 - 4ab + b^4\).
When a binomial involves subtraction, the sign will alternate. The first term will always be positive, and then the second term will be negative and the pattern will continue. So what if there is a coefficient with the terms. Then “a” becomes that term and “b” becomes the second term.

**Example:**

\[(3x - 2y)^4 = (3x)^4 + 4(3x)^3(-2y) + 6(3x)^2(-2y)^2 + 4(3x)(-2y)^3 + (-2y)^4 = \]

\[81x^4 + (-8y)(27x^3) + 6(9x^2)(4y^2) + 12x(-8y^3) + 16y^4\]

\[81x^4 - 216x^3y + 216x^2y^2 - 96xy^3 + 16y^4\]

**YOU TRY** Use Pascal’s Triangle to expand each of the following binomials. You may need to expand the triangle to answer a few of the questions.

1. \((2x + 3y)^3\)
2. \((a - b^2)^3\)
3. \((x^2 - 2y)^5\)

Hopefully you can see the advantage of using Pascal’s Triangle in expanding a binomial. But, what if you wanted to expand \((a + c)^{20}\). Figuring out what the exponents would be would be easy. We know we start with the first variable a to the 20th power and decrease it by one until we get to the second term, and then starting with the second term, we start with the c and increase the power to it by one until we get to the last term where we would have \(c^{20}\). Finding the coefficients, would be the hard part. That’s where the Binomial Theorem comes in. The Binomial Theorem, allows us to find the coefficients. The formal expression of the Binomial Theorem is \((a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\). The formula looks intimidating, but as you will see in just a minute, it’s not. The \(n\), represent the power that you are raising the binomial to. If we look at the last part of the expression, \(a^{n-k} b^k\) is exactly what you were doing before. The only new part of the expression is the \(\binom{n}{k}\). This is the piece that will give you the coefficient of each of the terms. So, what does \(\binom{n}{k}\) mean? \(\binom{n}{k} = \frac{n!}{(n-k)!k!}\). The exclamation does not mean that the variables are excited, they are known as factorials. The quickest way to see how this piece works is just to see how it works.
Let’s look at \((x+y)^4\). (We’re starting with something that you already know so you can see how the \(\binom{n}{k}\) actually works.)

We know that we can expand the variables, so we’re going to start there and leave space for the coefficients.

\[(x+y)^4 = \underline{\text{____}}x^4 + \underline{\text{____}}x^3y + \underline{\text{____}}x^2y^2 + \underline{\text{____}}xy^3 + \underline{\text{____}}y^4.\]

Since there is no constant with the initial \(x\) and \(y\) we know that the constants with the \(x^4\) and the \(y^4\) will be 1. So we only need to find three more constants.

To find the second term, we know that \(n = 4\) and \(k = 1\), why is \(k = 1\)? because we’re looking for the second coefficient. (Remember from above, \(k\) starts at 0.) So we’re going to find \(\binom{4}{1}\).

The factorial (!) means that we’re going to start with the number that we have and multiply all the numbers until we get all the way down to 1.

\[
\binom{4}{1} = \frac{4!}{(4-1)!!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)!!} = 4.
\]

Notice that we divided out (cancelled) the like terms and we were left with 4. You should remember from earlier that is the coefficient of the second term. Now we need to find the coefficient of the third term. If you remember one of the first things you read about Pascal’s Triangle was symmetric, so we already have the next term and the last term. So using the Binomial theorem, we just discovered, that

\[
(x+y)^4 = \underline{\text{____}}x^4 + \underline{\text{____}}x^3y + \underline{\text{____}}x^2y^2 + \underline{\text{____}}xy^3 + \underline{\text{____}}y^4.
\]

I know that was quick so let’s do another one. Rather than going horizontally we’ll go vertically so that we can do this step by step and we’ll put it all together at the end.

\[(x-y)^6 = \]

We’ll only look at the coefficients, since we know that the first variable will start with a power of 6 and decrease down and then starting with the second term we will add the \(y\) term and continue increasing the power until we get to the last term.

\[(x-y)^6 = \]

Since we’re going to the 6th power, there will be 7 terms and since there are no constants with the variables, we know the first and last coefficients will be one, so we do not need to do them.

<table>
<thead>
<tr>
<th>Term</th>
<th>Coefficient</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Coefficient</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Second Coefficient</td>
<td>(\frac{6}{1} = \frac{6!}{(6-1)!!})</td>
<td>(\frac{6}{1} = \frac{6!}{(6-1)!!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)!!} = 6)</td>
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</tbody>
</table>
Third Coefficient

\[
\binom{6}{2} = \frac{6!}{(6-2)!2!} = \frac{6!}{(6-2)!2!} = \frac{6!}{(6-2)!2!} = \frac{6!}{(6-2)!2!} = 15
\]

Fourth Coefficient

\[
\binom{6}{3} = \frac{6!}{(6-3)!3!} = \frac{6!}{(6-3)!3!} = \frac{6!}{(6-3)!3!} = \frac{6!}{(6-3)!3!} = 20
\]

Fifth Coefficient

\[
\binom{6}{4} = \frac{6!}{(6-4)!2!} = \frac{6!}{(6-4)!2!} = \frac{6!}{(6-4)!2!} = \frac{6!}{(6-4)!2!} = 15
\]

Sixth Coefficient

\[
\binom{6}{5} = \frac{6!}{(6-5)!2!} = \frac{6!}{(6-5)!2!} = \frac{6!}{(6-5)!2!} = \frac{6!}{(6-5)!2!} = 6
\]

Seventh Coefficient

\[
1
\]

So our final answer is \((x - y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6\) (Remember that we really didn’t need to do the 5th, 6th, or 7th coefficient since there are no coefficients with the variables in the initial problem and Pascal’s Triangle is symmetric and because we’re subtracting, the signs will switch from term to term in the final answer.

So let’s try another where there is a coefficient.

\((2x - 3y)^5\)

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<td>1</td>
</tr>
<tr>
<td>Second Coefficient</td>
<td>(\frac{5!}{(5-1)!1!})</td>
<td>(\frac{5!}{(5-1)!1!})</td>
</tr>
<tr>
<td>Third Coefficient</td>
<td>(\frac{5!}{(5-2)!2!})</td>
<td>(\frac{5!}{(5-2)!2!})</td>
</tr>
<tr>
<td>Fourth Coefficient</td>
<td>(\frac{5!}{(5-3)!3!})</td>
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</tr>
<tr>
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<td>(\frac{5!}{(5-4)!4!})</td>
<td>(\frac{5!}{(5-4)!4!})</td>
</tr>
<tr>
<td>Sixth Coefficient</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

\((a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\) Remember just like earlier \(2x = a\) and \(-3y = b\). So when we expand, a and b are replaced with \(2x\) and \(-3y\). So substituting the value of \(a\) and \(b\) back into the equation and simplifying, we will have the following.
\[(2x - 3y)^5 = 1(2x)^5 + 5(2x)^4(-3y) + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3 + 5(2x)(-3y)^4 + (-3y)^5\]
\[= 32x^5 - 15y(16x^4) + 10(8x^3)(9y^2) + 10(4x^2)(-27y^3) + 10x(81y^4) - 243y^5\]
\[= 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5\]

**Your Turn:** Use the binomial theorem to expand each of the following.

1. \((x + y)^7\)

2. \((a + b)^8\)

3. \((x - 2y)^6\)

4. \((3x - y)^6\)

\[(a + b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 = 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7\]